

Stochastic Integral with respect to Cylindrical Wiener Process

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Dedicated to Professor Dominik Szynal on the occasion of his 60-th birthday

Abstract

This paper is devoted to a construction of the stochastic Itô integral with respect to infinite dimensional cylindrical Wiener process. The construction given is an alternative one to that introduced by DaPrato and Zabczyk [3]. The connection of the introduced integral with the integral defined by Walsh [9] is provided as well.

1 Introduction

Recently there have been written several papers devoted to stochastic partial differential equations forced by cylindrical Wiener process, e.g., [4], [2] and [7]. In the study of stochastic partial differential equations some authors (see references given in Chapter 4 in [3]) have used a stochastic integral with respect to the so-called Brownian sheet, which is a special kind of cylindrical Wiener process, rather than with respect to cylindrical Wiener process in general form.

In the paper we provide a construction of stochastic integral with respect to an infinite dimensional cylindrical Wiener process alternative to the construction given by DaPrato and Zabczyk in their monograph [3]. We introduce the convenient construction which is

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based, by analogy to the construction given by Ichikawa [6] for the integral with respect to classical infinite dimensional Wiener process, on the stochastic integrals with respect to real-valued Wiener processes. The advantage of using of such a construction is that we can use basic results and arguments of the finite dimensional case. Finally, we compare the integral constructed in the paper with the integral introduced by Walsh [9].

Let us recall from [6] the definition of Wiener process with values in Hilbert space U (called later the classical infinite dimensional Wiener process) and the stochastic integral with respect to this Wiener process.

def1 **Definition 1** *Let $Q : U \rightarrow U$ be a linear symmetric non-negative nuclear operator ($\text{tr}Q < +\infty$). A square integrable U -valued stochastic process $W(t)$, $t \geq 0$, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where \mathcal{F}_t denote σ -fields such that $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for $t < s$, is called **Wiener process** with covariance operator Q if:*

1. $W(0) = 0$,
2. $EW(t) = 0$, $\text{Cov}[W(t) - W(s)] = (t - s)Q$ for all $s, t \geq 0$,
3. W has independent increments,
4. W has continuous trajectories,
5. W is adapted with respect to the filtration (\mathcal{F}_t) , that is, for any $t \geq 0$, $W(t)$ is \mathcal{F}_t -measurable.

In the light of the above, Wiener process is Gaussian and has the following structure: let $\{d_i\} \subset U$ be an orthonormal set of eigenvectors of Q with corresponding eigenvalues ζ_i (so $\text{tr}Q = \sum_{i=1}^{\infty} \zeta_i$), then $W(t) = \sum_{i=1}^{\infty} \beta_i(t)d_i$, where β_i are independent real Wiener processes with $E(\beta_i^2(t)) = \zeta_i t$. This type of structure of Wiener process will be used in definition of the stochastic integral.

Let $L(U, Y)$ denote the space of linear bounded operators from U into Y .

For any Hilbert space Y we denote by $M(Y)$ the space of all stochastic processes $g : [0, T] \times \Omega \rightarrow L(U, Y)$ such that

$$E \left(\int_0^T \|g(t)\|_{L(U, Y)}^2 dt \right) < +\infty$$

and for all $u \in U$, $g(t)u$ is a Y -valued stochastic process measurable with respect to the filtration (\mathcal{F}_t) .

The stochastic integral $\int_0^t g(s)dW(s) \in Y$ is defined for all $g \in M(Y)$ by

$$\int_0^t g(s)dW(s) = \lim_{m \rightarrow \infty} \sum_{i=1}^m \int_0^t g(s)d_i d\beta_i(s)$$

in $L^2(\Omega)$ sense.

We shall show that the series in the above formula is convergent.

Let $W^{(m)}(t) = \sum_{i=1}^m d_i \beta_i(t)$. Then, the integral

$$\int_0^t g(s)dW^{(m)}(s) = \sum_{i=1}^m \int_0^t g(s)d_i d\beta_i(s)$$

is well defined for $g \in M(Y)$ and additionally

$$\int_0^t g(s)dW^{(m)}(s) \xrightarrow{m \rightarrow \infty} \int_0^t g(s)dW(s) \quad \text{in } Y$$

in $L^2(\Omega)$ sense.

This convergence comes from the fact that the sequence

$$y_m = \int_0^t g(s)dW^{(m)}(s), \quad m \in \mathbb{N}$$

is Cauchy sequence in the space of square integrable random variables. For, using features of stochastic integrals with respect to $\beta_i(s)$, for any $m, n \in \mathbb{N}$, $m < n$, we have:

$$\begin{aligned} E(\|y_n - y_m\|_Y^2) &= \sum_{i=m+1}^n \zeta_i E \int_0^t (g(s)d_i, g(s)d_i)_Y ds \\ &\leq \left(\sum_{i=m+1}^n \zeta_i \right) E \int_0^t \|g(s)\|_{L(U,Y)}^2 ds \xrightarrow{m,n \rightarrow \infty} 0. \end{aligned} \tag{1}$$

Hence, there exists a limit of the sequence (y_m) which defines the stochastic integral $\int_0^t g(s)dW(s)$.

The above construction of the stochastic integral required the assumption that Q was a nuclear operator. (This assumption was used in (1).) However, it is possible to extend the definition of the stochastic integral to the case of general bounded self-adjoint, non-negative operator Q on Hilbert space U . (But it will require some restrictions on the integrand g .) Stochastic integral for this case has been defined e.g. in the monograph [3]. (To avoid trivial complications we shall assume that Q is strictly positive, that is: Q is non-negative and $Qx \neq 0$ for $x \neq 0$.)

Let us recall the following definition.

def2 **Definition 2** ([1] or [3]) Let E and F be separable Hilbert spaces with orthonormal bases $\{e_k\} \subset E$ and $\{f_j\} \subset F$, respectively. A linear bounded operator $T : E \rightarrow F$ is called **Hilbert-Schmidt operator** if $\sum_{k=1}^{\infty} \|Te_k\|_F^2 < +\infty$.

Because

$$\sum_{k=1}^{\infty} \|Te_k\|_F^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (Te_k, f_j)_F^2 = \sum_{j=1}^{\infty} \|T^*f_j\|_E^2,$$

where T^* denotes the operator adjoint to T , then the definition of Hilbert-Schmidt operator and the number $\|T\|_{HS} = (\sum_{k=1}^{\infty} \|Te_k\|_F^2)^{\frac{1}{2}}$ do not depend on the basis $\{e_k\}$, $k \in \mathbb{N}$. Moreover $\|T\|_{HS} = \|T^*\|_{HS}$.

Additionally, $L_2(E, F)$ – the set of all Hilbert-Schmidt operators from E into F , endowed with the norm $\|\cdot\|_{HS}$ defined above, is a separable Hilbert space.

Let us introduce the subspace U_0 of the space U defined by $U_0 = Q^{\frac{1}{2}}(U)$ with the norm

$$\|u\|_{U_0} = \|Q^{-\frac{1}{2}}u\|_U, \quad u \in U_0.$$

Assume that U_1 is an arbitrary Hilbert space such that U is continuously embedded into U_1 and the embedding of U_0 into U_1 is a Hilbert-Schmidt operator.

In particular

1. When $Q = I$, then $U_0 = U$ and the embedding of U into U_1 is Hilbert-Schmidt operator.
2. When Q is a nuclear operator, that is $\text{tr}Q < +\infty$, then $U_0 = Q^{\frac{1}{2}}(U)$ and we can take $U_1 = U$. Because in this case $Q^{\frac{1}{2}}$ is Hilbert-Schmidt operator then the embedding $U_0 \subset U$ is Hilbert-Schmidt operator.

2 Stochastic integral with respect to cylindrical Wiener process

We denote by $L_2^0 = L_2(U_0, Y)$ the space of Hilbert-Schmidt operators acting from U_0 into Y , and by $L = L(U, Y)$, like earlier, we denote the space of linear bounded operators from U into Y .

Let us consider the norm of the operator $\psi \in L_2^0$:

$$\begin{aligned}
\|\psi\|_{L_2^0}^2 &= \sum_{h,k=1}^{\infty} (\psi g_h, f_k)_Y^2 = \sum_{h,k=1}^{\infty} \lambda_h (\psi e_h, f_k)_Y^2 \\
&= \|\psi Q^{\frac{1}{2}}\|_{HS}^2 = \text{tr}(\psi Q \psi^*),
\end{aligned}$$

where $g_j = \sqrt{\lambda_j} e_j$, and $\{\lambda_j\}$, $\{e_j\}$ are eigenvalues and eigenfunctions of the operator Q ; $\{g_j\}$, $\{e_j\}$ and $\{f_j\}$ are orthonormal bases of spaces U_0 , U and Y , respectively.

The space L_2^0 is a separable Hilbert space with the norm $\|\psi\|_{L_2^0}^2 = \text{tr}(\psi Q \psi^*)$.

In particular

1. When $Q = I$ then $U_0 = U$ and the space L_2^0 becomes $L_2(U, Y)$.
2. When Q is a nuclear operator, that is $\text{tr} Q < +\infty$, then $L(U, Y) \subset L_2(U_0, Y)$. For, assume that $K \in L(U, Y)$ that is K is linear bounded operator from the space U into Y . Let us consider the operator $\psi = K|_{U_0}$, that is the restriction of operator K to the space U_0 , where $U_0 = Q^{\frac{1}{2}}(U)$. Because Q is nuclear operator, then $Q^{\frac{1}{2}}$ is Hilbert-Schmidt operator. So, the embedding J of the space U_0 into U is Hilbert-Schmidt operator. We have to compute the norm $\|\psi\|_{L_2^0}^2$ of the operator $\psi : U_0 \rightarrow Y$. We obtain $\|\psi\|_{L_2^0}^2 \equiv \|KJ\|_{L_2^0}^2 = \text{tr} KJ(KJ)^*$, where $J : U_0 \rightarrow U$.

Because J is Hilbert-Schmidt operator and K is linear bounded operator then, basing on the theory of Hilbert-Schmidt operators (e.g. [5], Chapter I), KJ is Hilbert-Schmidt operator, too. Next, $(KJ)^*$ is Hilbert-Schmidt operator. In consequence, $KJ(KJ)^*$ is nuclear operator, so $\text{tr} KJ(KJ)^* < +\infty$. Hence, $\psi = K|_{U_0}$ is Hilbert-Schmidt operator on the space U_0 , that is $K \in L_2(U_0, Y)$.

Let $\{g_j\}$ denote an orthonormal basis in U_0 and $\{\beta_j\}$ be a family of independent standard real-valued Wiener processes.

Although Propositions 1. and 2. introduced below are known (see, e.g. Proposition 4.11 in the monograph [3]), because of their importance we formulate them again and provide with detailed proofs.

prop1 **Proposition 1** *The formula*

$$W_c(t) = \sum_{j=1}^{\infty} g_j \beta_j(t), \quad t \geq 0 \tag{2}$$

defines Wiener process in U_1 with covariance operator Q_1 such that $\text{tr} Q_1 < +\infty$.

Proof: This comes from the fact that the series (2) is convergent in space $L^2(\Omega, \mathcal{F}, P; U_1)$.

We have

$$\begin{aligned}
& E \left(\left\| \sum_{j=1}^n g_j \beta_j(t) - \sum_{j=1}^m g_j \beta_j(t) \right\|_{U_1}^2 \right) = E \left(\left\| \sum_{j=m+1}^n g_j \beta_j(t) \right\|_{U_1}^2 \right) = \\
& = E \left(\sum_{j=m+1}^n g_j \beta_j(t), \sum_{k=m+1}^n g_k \beta_k(t) \right)_{U_1} = E \sum_{j=m+1}^n (g_j \beta_j(t), g_j \beta_j(t))_{U_1} \\
& = E \left(\sum_{j=m+1}^n (g_j, g_j)_{U_1} \beta_j^2(t) \right) = t \sum_{j=m+1}^n \|g_j\|_{U_1}^2, \quad n \geq m \geq 1.
\end{aligned}$$

From the assumption, the embedding $J : U_0 \rightarrow U_1$ is Hilbert–Schmidt operator, then for the basis $\{g_j\}$, complete and orthonormal in U_0 , we have $\sum_{j=1}^{\infty} \|Jg_j\|_{U_1}^2 < +\infty$. Because $Jg_j = g_j$ for any $g_j \in U_0$, then $\sum_{j=1}^{\infty} \|g_j\|_{U_1}^2 < +\infty$ which means $\sum_{j=m+1}^n \|g_j\|_{U_1}^2 \rightarrow 0$ when $m, n \rightarrow \infty$.

Conditions 1), 2), 3) and 5) of the definition of Wiener process are obviously satisfied. The process defined by (2) is Gaussian because $\beta_j(t)$, $j \in \mathbb{N}$, are independent Gaussian processes. By Kolmogorov test theorem (see, e.g. [3], Theorem 3.3), trajectories of the process $W_c(t)$ are continuous (condition 4) of the definition of Wiener process) because $W_c(t)$ is Gaussian.

Let $Q_1 : U_1 \rightarrow U_1$ denote the covariance operator of the process $W_c(t)$ defined by (2). From the definition of covariance, for $a, b \in U_1$ we have:

$$\begin{aligned}
(Q_1 a, b)_{U_1} &= E(a, W_c(t))_{U_1} (b, W_c(t))_{U_1} = E \left(\sum_{j=1}^{\infty} (a, g_j)_{U_1} (b, g_j)_{U_1} \beta_j^2(t) \right) \\
&= t \sum_{j=1}^{\infty} (a, g_j)_{U_1} (b, g_j)_{U_1} = t \left(\sum_{j=1}^{\infty} g_j (a, g_j)_{U_1}, b \right)_{U_1}.
\end{aligned}$$

Hence $Q_1 a = t \sum_{j=1}^{\infty} g_j (a, g_j)_{U_1}$.

Because the covariance operator Q_1 is non-negative, then (by Proposition C.3 in [3]) Q_1 is a nuclear operator if and only if $\sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} < +\infty$, where $\{h_j\}$ is an orthonormal basis in U_1 .

From the above considerations

$$\sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} \leq t \sum_{j=1}^{\infty} \|g_j\|_{U_1}^2 \quad \text{and then} \quad \sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} \equiv \text{tr} Q_1 < +\infty.$$

□

prop2 **Proposition 2** For any $a \in U$ the process

$$(a, W_c(t))_U = \sum_{j=1}^{\infty} (a, g_j)_U \beta_j(t) \quad (3)$$

is real-valued Wiener process and

$$E (a, W_c(t))_U (b, W_c(s))_U = (t \wedge s) (Qa, b)_U \quad \text{for } a, b \in U.$$

Additionally, $\text{Im } Q_1^{\frac{1}{2}} = U_0$ and $\|u\|_{U_0} = \|Q_1^{-\frac{1}{2}}u\|_{U_1}$.

Proof: We shall prove that the series (3) defining the process $(a, W_c(t))_U$ is convergent in the space $L^2(\Omega, \mathcal{F}, P)$.

Let us notice that the series (3) is the sum of independent random variables with zero mean. Then the series does converge in $L^2(\Omega, \mathcal{F}, P)$ if and only if the following series $\sum_{j=1}^{\infty} E ((a, g_j)_U \beta_j(t))^2$ converges.

Because J is Hilbert–Schmidt operator, we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} E ((a, g_j)_U \beta_j(t))^2 &= \sum_{j=1}^{\infty} (a, g_j)_U^2 \leq \|a\|_U^2 \sum_{j=1}^{\infty} \|g_j\|_U^2 \\ &\leq C \|a\|_U^2 \sum_{j=1}^{\infty} \|Jg_j\|_{U_1}^2 < +\infty. \end{aligned}$$

Hence, the series (3) does converge. Moreover, when $t \geq s \geq 0$, we have

$$\begin{aligned} E ((a, W_c(t))_U (b, W_c(s))_U) &= E ((a, W_c(t) - W_c(s))_U (b, W_c(s))_U) \\ &+ E ((a, W_c(s))_U (b, W_c(s))_U) \\ &= E ((a, W_c(s))_U (b, W_c(s))_U) \\ &= E \left(\left[\sum_{j=1}^{\infty} (a, g_j)_U \beta_j(s) \right] \left[\sum_{k=1}^{\infty} (b, g_k)_U \beta_k(s) \right] \right). \end{aligned}$$

Let us introduce

$$S^a := \sum_{j=1}^{\infty} (a, g_j)_U \beta_j(t), \quad S^b := \sum_{k=1}^{\infty} (b, g_k)_U \beta_k(t), \quad \text{for } a, b \in U.$$

Next, let S_N^a and S_N^b denote the partial sums of the series S^a and S^b , respectively. From the above considerations the series S^a and S^b are convergent in $L^2(\Omega, \mathcal{F}, P; \mathbb{R})$. Hence

$E(S^a S^b) = \lim_{N \rightarrow \infty} E(S_N^a S_N^b)$. In fact,

$$\begin{aligned} E|S_N^a S_N^b - S^a S^b| &= E|S_N^a S_N^b - S_n^a S^b + S^b S_n^a - S^b S^a| \\ &\leq E|S_N^a| |S_N^b - S^b| + E|S^b| |S_N^a - S^a| \\ &\leq (E|S_N^a|^2)^{\frac{1}{2}} (E|S_N^b - S^b|^2)^{\frac{1}{2}} \\ &\quad + (E|S^b|^2)^{\frac{1}{2}} (E|S_N^a - S^a|^2)^{\frac{1}{2}} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

because S_N^a converges to S^a and S_N^b converges to S^b in quadratic mean.

Additionally, $E(S_N^a S_N^b) = t \sum_{j=1}^N (a, g_j)_U (b, g_j)_U$ and when $N \rightarrow +\infty$

$$E(S^a S^b) = t \sum_{j=1}^{\infty} (a, g_j)_U (b, g_j)_U.$$

Let us notice that

$$\begin{aligned} (Q_1 a, b)_{U_1} &= E(a, W_c(1))_{U_1} (b, W_c(1))_{U_1} = \sum_{j=1}^{\infty} (a, g_j)_{U_1} (b, g_j)_{U_1} \\ &= \sum_{j=1}^{\infty} (a, Jg_j)_{U_1} (b, Jg_j)_{U_1} = \sum_{j=1}^{\infty} (J^* a, g_j)_{U_0} (J^* b, g_j)_{U_0} \\ &= \left(J^* a, \sum_{j=1}^{\infty} (J^* b, g_j) g_j \right)_{U_0} = (J^* a, J^* b)_{U_0} = (JJ^* a, b)_{U_1}. \end{aligned}$$

That gives $Q_1 = JJ^*$. In particular

$$\left\| Q_1^{\frac{1}{2}} a \right\|_{U_1}^2 = (JJ^* a, a)_{U_1} = \|J^* a\|_{U_0}^2, \quad a \in U_1. \quad (4)$$

Having (4), we can use theorems about images of linear operators (e.g., [3], Appendix B.2, Proposition B.1 (ii)).

By this theorem $\text{Im } Q_1^{\frac{1}{2}} = \text{Im } J$. But for any $j \in \mathbb{N}$, and $g_j \in U_0$, $Jg_j = g_j$, that is $\text{Im } J = U_0$. Then $\text{Im } Q_1^{\frac{1}{2}} = U_0$.

Moreover, the operator $G = Q_1^{-\frac{1}{2}} J$ is a bounded operator from U_0 on U_1 . From (4) the joint operator $G^* = J^* Q_1^{-\frac{1}{2}}$ is an isometry, so G is isometry, too. Then

$$\left\| Q_1^{-\frac{1}{2}} u \right\|_{U_1} = \left\| Q_1^{-\frac{1}{2}} J u \right\|_{U_1} = \|u\|_{U_0}.$$

□

In the case when Q is nuclear operator, $Q^{\frac{1}{2}}$ is Hilbert-Schmidt operator. Taking $U_1 = U$, the process $W_c(t)$, $t \geq 0$, defined by (2) is the classical Wiener process introduced in Definition 1.

def3 **Definition 3** *The process $W_c(t)$, $t \geq 0$, defined in (2), is called cylindrical Wiener process in U when $\text{tr}Q = +\infty$.*

The stochastic integral with respect to cylindrical Wiener process is defined as follows.

As we have already written above, the process $W_c(t)$ defined by (2) is a Wiener process in the space U_1 with the covariance operator Q_1 such that $\text{tr}Q_1 < +\infty$. Then the stochastic integral $\int_0^t g(s)dW_c(s) \in Y$, where $g(s) \in L(U_1, Y)$, with respect to the Wiener process $W_c(t)$ is well defined on U_1 .

Let us notice that U_1 is not uniquely determined. The space U_1 can be an arbitrary Hilbert space such that U is continuously embedded into U_1 and the embedding of U_0 into U_1 is a Hilbert-Schmidt operator. We would like to define the stochastic integral with respect to cylindrical Wiener process $W_c(t)$ (given by (2)) in such a way that the integral is well defined on the space U and does not depend on the choice of the space U_1 .

We denote by $N(Y)$ the space of all stochastic processes

$$\Phi : [0, T] \times \Omega \rightarrow L_2(U_0, Y) \quad (5)$$

such that

$$E \left(\int_0^T \|\Phi(t)\|_{L_2(U_0, Y)}^2 dt \right) < +\infty \quad (6)$$

and for all $u \in U_0$, $\Phi(t)u$ is a Y -valued stochastic process measurable with respect to the filtration (\mathcal{F}_t) .

The stochastic integral $\int_0^t \Phi(s)dW_c(s) \in Y$ with respect to cylindrical Wiener process, given by (2) for any process $\Phi \in N(Y)$, can be defined as the limit

$$\int_0^t \Phi(s)dW_c(s) = \lim_{m \rightarrow \infty} \sum_{j=1}^m \int_0^t \Phi(s)g_j d\beta_j(s) \quad \text{in } Y \quad (7)$$

in $L^2(\Omega)$ sense.

Comment: Before we prove that the stochastic integral given by the formula (7) is well defined, let us recall properties of the operator Q_1 . From Proposition 1, cylindrical Wiener process $W_c(t)$ given by (2) has the covariance operator $Q_1 : U_1 \rightarrow U_1$, which is a nuclear operator in the space U_1 , that is $\text{tr}Q_1 < +\infty$. Next, basing on Proposition 2, $Q_1^{\frac{1}{2}} : U_1 \rightarrow U_0$, $\text{Im } Q_1^{\frac{1}{2}} = U_0$ and $\|u\|_{U_0} = \|Q_1^{-\frac{1}{2}}u\|_{U_1}$ for $u \in U_0$.

Moreover, from the above considerations and properties of the operator Q_1 we may deduce that $L(U_1, Y) \subset L_2(U_0, Y)$. This means that each operator $\Phi \in L(U_1, Y)$, that is linear and bounded from U_1 into Y , is Hilbert-Schmidt operator acting from U_0 into Y ,

that is $\Phi \in L_2(U_0, Y)$ when $\text{tr} Q_1 < +\infty$ in U_1 . This means that conditions (5) and (6) for the family $N(Y)$ of integrands are natural assumptions for the stochastic integral given by (7).

Now, we shall prove that the series from the right hand side of (7) is convergent.

Denote

$$W_c^{(m)}(t) := \sum_{j=1}^m g_j \beta_j(t)$$

and

$$Z_m := \int_0^t \Phi(s) W_c^{(m)}(s), \quad t \in [0, T].$$

Then, we have

$$\begin{aligned} E(\|Z_n - Z_m\|_Y^2) &= E \left\| \sum_{j=m+1}^n \int_0^t \Phi(s) g_j d\beta_j(s) \right\|_Y^2 \quad \text{for } n \geq m \geq 1 \\ &\leq E \sum_{j=m+1}^n \int_0^t \|\Phi(s) g_j\|_Y^2 ds \xrightarrow{m, n \rightarrow \infty} 0, \end{aligned}$$

because from the assumption (6)

$$E \int_0^t \left(\sum_{j=1}^{\infty} \|\Phi(s) g_j\|_Y^2 \right) ds < +\infty.$$

Then, the sequence (Z_m) is Cauchy sequence in the space of square-integrable random variables. So, the stochastic integral with respect to cylindrical Wiener process given by (7) is well defined.

As we have already mentioned, the space U_1 is not uniquely determined. Hence, the cylindrical Wiener process $W_c(t)$ defined by (2) is not uniquely determined either.

Let us notice that the stochastic integral defined by (7) does not depend on the choice of the space U_1 . Firstly, in the formula (7) there are not elements of the space U_1 but only $\{g_j\}$ -basis of U_0 . Additionally, in (7) there are not eigenfunctions of the covariance operator Q_1 . Secondly, the class $N(Y)$ of integrands does not depend on the choice of the space U_1 because (by Proposition 2.) the spaces $Q_1^{\frac{1}{2}}(U_1)$ are identical for any spaces U_1 :

$$Q_1^{\frac{1}{2}} : U_1 \rightarrow U_0 \quad \text{and} \quad \text{Im } Q_1^{\frac{1}{2}} = U_0.$$

Hence, the stochastic integral with respect to infinite dimensional Wiener process, even cylindrical, can be obtained in the above sense as the limit of stochastic integrals with respect to real-valued Wiener processes.

3 Connection with Walsh integral

In this section we compare the integral defined in the previous section with the integral constructed by Walsh [9].

Let us recall from [9] the necessary definitions. Assume that (E, \mathcal{E}) is a Lusin space, i.e. a measurable space homeomorphic to a Borel subset of the line. (Let us notice that this space includes all Euclidean spaces and, more generally, all Polish spaces.) Suppose $\mathcal{A} \subset \mathcal{E}$ is an algebra.

def4 **Definition 4** Let (\mathcal{F}_t) be a right continuous filtration. A process $\{M_t(A), \mathcal{F}_t, t \geq 1, A \in \mathcal{A}\}$ is a **martingale measure** if

1. $M_0(A) = 0$,
2. if $t > 0$, M_t is a σ -finite L^2 -valued measure,
3. $\{M_t(A), \mathcal{F}_t, t \geq 0\}$ is a martingale.

def5 **Definition 5** A martingale measure M is **orthogonal** if, for any two disjoint sets A and B in \mathcal{A} , the martingales $\{M_t(A), \mathcal{F}_t, t \geq 1\}$ and $\{M_t(B), \mathcal{F}_t, t \geq 1\}$ are orthogonal.

Let us notice that an example of an orthogonal martingale measure is a white noise. If W is a white noise on $E \times \mathbb{R}_+$, define $M_t(A) = W(A \times [0, t])$. This is clearly martingale measure, and if $A \cap B = \emptyset$, $M_t(A)$ and $M_t(B)$ are independent, hence orthogonal.

We know how to integrate over dx for fixed t – this is the Bochner integral – and over dt for fixed sets A – this is the Itô integral. The problem is to integrate over dx and dt at the same time. Unfortunately, it is not possible to construct a stochastic integral with respect to all martingale measures. We shall add some conditions and define a new class of martingale measures.

def6 **Definition 6** A martingale measure M is **worthy** if there exists a random σ -finite measure $K(\Lambda, \omega)$, $\Lambda \in \mathcal{E} \times \mathcal{E} \times \mathcal{B}$, where \mathcal{B} is Borel sets on \mathbb{R}_+ , $\omega \in \Omega$, such that

1. K is positive definite and symmetric in x and y ,
2. for fixed A, B , $\{K(A \times B \times (0, t]), t \geq 0\}$ is predictable,
3. for all $n \in \mathbb{N}$, $\mathbb{E} \{K(E_n \times E_n \times [0, T])\} < +\infty$, where $E_n \subset \mathcal{E}$,
4. for any rectangle Λ , $|M(\Lambda)| \leq K(\Lambda)$.

(We call K the **dominating measure** of M .)

Let us notice, that conditions of the above definition are satisfied by orthogonal martingale measures, that is orthogonal martingale measures are worthy.

As usual, we first define the integral for elementary functions, then for simple functions, and then for all functions in a certain class.

def7 **Definition 7** *function f is elementary if it is of the form*

$$f(s, x, w) = I_{(a,b]}(s)I_A(x)X(w), \quad (8)$$

where $0 \leq a \leq t$, X is bounded and \mathcal{F} – measurable, and $A \in \mathcal{E}$.

A function f is **simple** if it is a finite sum of elementary functions.

We shall denote the class of simple functions by \mathcal{S} .

def8 **Definition 8** *The predictable σ -field \mathcal{P} on $\Omega \times E \times \mathbb{R}_+$ is the σ -field generated by \mathcal{S} . A function is **predictable** if it is \mathcal{P} -measurable.*

We define a norm $\|\cdot\|_M$ on the predictable functions by

$$\|f\|_M = \mathbb{E} \{(|f|, |f|)_K\}^{\frac{1}{2}},$$

where

$$(f, g)_K = \int_{E \times E \times \mathbb{R}_+} f(s, x)g(s, y)K(dx dy ds).$$

Let \mathcal{P}_M be the class of all predictable f for which $\|f\|_M < +\infty$.

prop3 **Proposition 3** *The class \mathcal{P}_M is a Banach space. Moreover, \mathcal{S} is dense in \mathcal{P}_M .*

(For proof and details, see [9].)

Now, we can follow Walsh and define stochastic integral as a martingale measure.

If f is an elementary function, that is f has the form (8), define a martingale measure $f \cdot M$ by

$$f \cdot M_t(B) \stackrel{\text{df}}{=} X(w) [M_{t \wedge b}(A \cap B) - M_{t \wedge a}(A \cap B)].$$

prop4 **Proposition 4** *(Lemma 2.4, [9]) The martingale measure $f \cdot M$ is worthy. Moreover*

$$\mathbb{E} \{(f \cdot M_t(B))^2\} \leq \|f\|_M^2 \quad \text{for all } B \in \mathcal{E}, t \leq T. \quad (9)$$

Now, we can define $f \cdot M$ for $f \in \mathcal{S}$ by linearity.

Suppose that $f \in \mathcal{P}_M$. By Proposition 3 there exist $f_n \in \mathcal{S}$ such that $\|f - f_n\|_M \rightarrow 0$ when $n \rightarrow \infty$. By (9), if $A \in \mathcal{E}$ and $t \leq T$,

$$\mathbb{E} \{ (f_m \cdot M_t(A) - f_n \cdot M_t(A))^2 \} \leq \|f_m - f_n\|_M \rightarrow 0, \text{ when } m, n \rightarrow \infty.$$

It follows that $(f_n \cdot M_t(a))$ is Cauchy sequence in $L^2(\Omega, \mathcal{F}, P)$, then it converges in L^2 to a martingale which we shall denote by $f \cdot M_t(A)$. Additionally, the limit is independent of the choice of the sequence (f_n) .

prop5

Proposition 5 (*Theorem 2.5, [9]*) *If $f \in \mathcal{P}_M$, then $f \cdot M$ is a worthy martingale measure.*

Now, because the stochastic integral is defined as a martingale measure, we define the "usual" stochastic integrals by

$$\int_0^t \int_A f(s, x) M(dx ds) = f \cdot M_t(A)$$

and

$$\int_0^t \int_E f(s, x) M(dx ds) = f \cdot M_t(E).$$

Let us consider the integral constructed by Walsh and recalled in this section in the case when $E \equiv \mathbb{R}^d$ and $M_t(A)$ is cylindrical Wiener process. In this case

$$M_t(A) \equiv W(t, A, w),$$

where $t \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$ with $mes(A) < +\infty$, and $w \in \Omega$. (In the remaining part of the paper we shall omit the argument w .)

For any A fixed, $\{W(\cdot, A)\}$ is a real-valued Wiener process, adapted to the filtration (\mathcal{F}_t) which does not depend on A . Moreover, $\mathbb{E}(W(t, A)W(s, B)) = t \wedge s \, mes(A \cap B)$.

Assume that $\phi : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$.

Using Walsh approach we can define the integral

$$J(\phi) := \int_0^T \int_{\mathbb{R}^d} \phi(t, x) W(dt, dx).$$

We start from simple functions of the form $\phi(t, x, w) = f(t, w)I_A(x)$. Then we have

$$J(\phi) = \int_0^T f(t) W(dt, A) \equiv \int_0^T f(t) dW(t, A).$$

Let us introduce the following classes of functions.

By P_T we denote the class of functions $\phi : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ satisfying the following conditions:

1. ϕ is measurable,
2. the function $\phi(t, x)$ is \mathcal{F}_t -measurable,
3. $\mathbb{E} \left(\int_0^T \int_{\mathbb{R}^d} |\phi(t, x)|^2 dt dx \right) < +\infty$.

By \tilde{P}_T we denote the class of functions $\phi : [0, T] \times \Omega \rightarrow L^2(\mathbb{R}^d)$ such that:

1. ϕ is measurable,
2. the function $\phi(t)$ is \mathcal{F}_t -measurable,
3. $\mathbb{E} \left(\int_0^T |\phi(t)|_{L^2(\mathbb{R}^d)}^2 dt \right) < +\infty$.

Let us notice that the both classes P_T and \tilde{P}_T coincide.

Now, we can formulate the following result.

prop6 **Proposition 6** *Let $W(t)$ be a cylindrical Wiener process. Then*

$$\int_0^T \phi(t) dW(t) = \int_0^T \int_{\mathbb{R}^d} \phi(t)(x) W(dt, dx) \quad (10)$$

for any $\phi \in \tilde{P}_T$.

Proof: It is enough to check the formula (10) for simple function $\phi(t)(x) \equiv f(t)I_A(x)$.

Let $\{e_k\}$, $k = 1, 2, \dots, d$, be a basis in \mathbb{R}^d , where $e_k = I_A / \text{mes}(A)$ and e_k , for $k = 2, \dots, d$, are arbitrary.

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \phi(t)(x) W(dt, dx) &\equiv \int_0^T \int_{\mathbb{R}^d} f(t) I_A(x) W(dt, dx) \\ &\equiv \sum_{k=1}^d \int_0^T (f(t), e_k) W(dt, e_k) \\ &\equiv \sum_{k=1}^d \int_0^T (f(t), e_k) dW(t)[e_k] = \int_0^T \phi(t) dW(t). \end{aligned}$$

□

Comment: Another, very recent example of integrating over random measures in multi-dimensional spaces has been given in the paper of Peszat and Zabczyk [8]. In the paper,

the noise is supposed to be a spatially homogeneous Wiener process in some special space. The authors describe its reproducing kernel and provide the concept of stochastic integral with respect to introduced Wiener process.

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